PROBABILITY

Exercise sheet 0: Counting and decisions

Counting.

Exercise 1. *Let us define the sets of permutations, combinations and variations.*

- *For each* $n \in \mathbb{N}^1$, *let* $\mathcal{P}_n$ *be the set of permutations of* $\{1, \ldots, n\}$, *i.e. the set of bijections of* $\{1, \ldots, n\}$ *to itself.*
- *For each* $k, n \in \mathbb{N}$ *such that* $1 \leq k \leq n$, *let* $\mathcal{C}_k^n$ *be the set of of combinations, i.e. the set of subsets of* $\{1, \ldots, n\}$ *of cardinality* k.
- *For each* $k, n \in \mathbb{N}$ *such that* $1 \leq k \leq n$ *let* $\mathcal{V}_k^n$ *be the set of variations, i.e. the set of sequences* $(j_1, \ldots, j_k)$ *of pairwise distinct elements of* $\{1, \ldots, n\}$.
- *Finally, for each* $k, n \in \mathbb{N}$, *let* $\mathcal{T}_k^n = \{1, \ldots, n\}^k$, *i.e.the set of sequences* $(j_1, \ldots, j_k)$ *of elements of* $\{1, \ldots, n\}$.

By finding natural mappings from $\mathcal{P}_n$ *to* $\mathcal{V}_k^n$ *and from* $\mathcal{V}_k^n$ *to* $\mathcal{C}_k^n$ *or otherwise, derive formulas for the cardinalities* $|\mathcal{P}_n|$, $\binom{n}{k} := C_k^n = |\mathcal{C}_k^n|$, $V_k^n := |\mathcal{V}_k^n|$ *and* $|\mathcal{T}_n| = |\{1, \ldots, n\}^k|$.

Exercise 2 (Simple walks). *A simple walk of* n *steps is a* $\mathbb{R}$*-valued sequence* $S_0, S_1, S_2, \ldots, S_n$ *such that* $S_0 = 0$ *and* $|S_i - S_{i-1}| = 1$. *How many different walks of length* n *are there? How many walks are there such that* $S_n = 0$?

Exercise 3 (Inspired by [1], Section 2.8, exercise 2.10). *How many different words can you form by scrambling the letters of each of the following words?*

- *"bikes";*
- *"label";*
- *"minimum".*

Exercise 4 (Inspired by [1], Section 2.8, exercise 2.13). *We wish to establish the following equality.*

$$\sum_{j=1}^{n} \binom{n}{j} \binom{j}{i} = \binom{n}{i} 2^{n-i}$$

We will do so in two different ways.

- *Double counting. Consider a group of* n *people. From these* n *people we want to form a committee of* $j \leq n$ *people. Further, within this committee we wish to choose* $i \leq j$ *different people to fill* i *different roles. Find two ways of computing the number of different ways of forming such a committee.*
- *Calculus. Apply Newton's formula to* $(1 + X)^n$ *and differentiate in* X.

[1] $\mathbb{N} = \{1, 2, \ldots\}$

Decisions.

Exercise 5 (Not so Black Jack)**.** *Suppose you are playing the following version of black jack: you can choose yourself what number you want to obtain as the sum of two cards and then your task is to obtain it with two cards.*

What number should you pick? [2]

Exercise 6 (Bets with knights)**.** *Probability was born with games of chance. One of the oldest recorded betting strategies was by Chevalier de Méré* [3]*, who offered the following bet: he throws four dice and if he gets at least one six, then he wins a piece of gold, otherwise you win. Should you take this bet?*

His other bet was as follows - he throws 24 times a pair of dice. If he gets at least once a double six, he wins. Should you accept this bet?

Exercise 7 (The Monty Hall problem)**.** *On your quest to save your friend, you encounter a sorcerer, guarding three doors.*

He tells you that your friend awaits behind one of the doors while behind each of the two other doors lies a fierce dragon. The sorcerer asks you to choose a door among the three. But before you open it, he points to one of the two remaining doors, and tells you that there is a dragon behind it. Given this information, he asks if you wish to reconsider your choice. While sorcerers are notorious for playing games with travelers like you, they are always true to their word. You can be sure he is not lying.

You can now choose between the door you chose initially and the remaining third door. What should you do?

References

[1] Conus D., Dalang R., *Introduction à la théorie des probabilités*. Presses polytechniques et universitaires romandes, 2015

[2] The Jack, Queen, King all equal to 10; for simplicity we choose the Ace always to be 1.

[3] with the real name Antoine Gombaud